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LETTER TO THE EDITOR

Quasi-exactly solvable models with an inhomogeneous magnetic field

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Abstract. Let group generators having finite-dimensional representation be realized as Hermitian linear differential operators without inhomogeneous terms as takes place, for example, for the $SO(n)$ group. Then corresponding group Hamiltonians containing terms linear in generators (along with quadratic ones) give rise to quasi-exactly solvable models with a magnetic field in a curved space. In particular, for the $SO(4)$ group Hamiltonian with isotropic quadratic part, the manifold within which a quantum particle moves has the geometry of the Einstein universe.

Quantum mechanical models admitting exact solutions of the Schrödinger equation attract quite understandable attention. Until recently it was considered to be axiomatic that there are only two possibilities: a model is exactly solvable or non-solvable. It turned out that there is also the third possibility—so-called quasi-exactly solvable models (QESM). In the whole space of states of QESM there exist finite-dimensional subspaces for which solving the Schrödinger equation is reduced to algebraic procedure, so exact solutions can be found in the algebraic form at least for a part of the spectrum. Firstly, QESM were found empirically [1, 2]. Later, it was shown that in the one-dimensional case they are intimately connected with $SU(2)$ algebra [3–5] of linear differential operators. The general procedure for obtaining QESM with the help of $SU(2)$ group is developed in [6] (see also reviews [7–9]).

QESM can be either one-dimensional or many-dimensional. In the latter case the manifold on which the wave function is defined is, in general, curved Riemannian space [10–12]. As a result, at least two factors affect the motion of a quantum particle within such a manifold: the metric (effective 'gravitational' field) and a usual scalar potential. Meanwhile, it is of interest to take into consideration one more factor—a magnetic field or its n-dimensional analogue (for brevity we will speak simply about a magnetic field).

Following the general procedure of obtaining QESM described in the references cited above, consider group Hamiltonian

$$H = C_{ab}L^aL^b + C_aL^a. \tag{1}$$

Here L^a are group generators obeying commutation relations for some Lie algebra. It is assumed that all coefficients in (1) are real and $C_{ab} = C_{ba}$.

In this letter a general approach is developed which enables one to construct QESM with a magnetic field. This goal is achieved by generalization of the approach of [13] (section 4). It was shown in [13] that $C_a = 0$ leads to both hermiticity of (1) with a certain measure in Riemannian space and an absence of a magnetic field. We will see that taking $C_a \neq 0$

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also preserves hermiticity and corresponds to an appearance of a certain magnetic field in the Schrödinger operator.

By construction, it is assumed that operators L^a in (1) can be realized in terms of linear differential operators and that the corresponding algebra admits finite-dimensional representations. From the very beginning, we restrict ourselves to operators having the form

$$L^a = i h^{a\mu} \frac{\partial}{\partial x^\mu} \tag{2}$$

without inhomogeneous terms, coefficients $h^{a\mu}$ are real. This form is valid for the $SO(n)$ group whose generators act in R^{n-1} (for $n = 3$ it gives us the well known operators of angular orbital momentum). In what follows we will not specify the explicit form of coefficients $h^{a\mu}$.

Substituting (2) into the Schrödinger equation

$$H\phi = E\phi. \tag{3}$$

The equation takes the form

$$-g^{\mu\nu} \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} + T^\mu \frac{\partial \phi}{\partial x^\mu} = E\phi. \tag{4}$$

Here

$$g^{\mu\nu} = C_{ab} h^{a\mu} h^{b\nu} \tag{5}$$

$$T^\mu = T_1^\mu + i T_2^\mu \qquad T_1^\mu = -C_{ab} h^{a\nu} h^{b\mu}_{,\nu} \qquad T_2^\mu = h^{a\mu} C_a. \tag{6}$$

Equation (4) can be rewritten as follows:

$$-g^{\mu\nu}(\nabla_\mu - A_\mu)(\nabla_\nu - A_\nu)\phi + U\phi = E\phi \tag{7}$$

where $g^{\mu\nu}$ can be viewed as contravariant components of the metric in the curved space, $g_{\mu\nu}$ being its covariant components, ∇_μ is the covariant derivative in this space.

Take into account that

$$\begin{aligned} g^{\mu\nu} \nabla_\mu \nabla_\nu \phi &= \frac{(\sqrt{g} g^{\mu\nu} \phi_{,\nu})_{,\mu}}{\sqrt{g}} \\ g &= \det g_{\mu\nu}. \end{aligned} \tag{8}$$

Comparing (4) and (7) one can easily obtain

$$U = g^{\mu\nu} A_\mu A_\nu - \frac{(\sqrt{g} A^\mu)_{,\mu}}{\sqrt{g}} \tag{9}$$

$$A^\mu = g^{\mu\nu} A_\nu = \frac{T^\mu + P^\mu}{2} \qquad P^\mu = \frac{(\sqrt{g} g^{\mu\nu})_{,\nu}}{\sqrt{g}}. \tag{10}$$

In a general case (7) does not have the Schrödinger form because of the presence of terms with A_μ. Now let, in the expression for them,

$$A_\mu = a_\mu + i b_\mu$$
$$a_\mu = \frac{T_{-1\mu} + P_\mu}{2} \qquad b_\mu = \frac{T_{2\mu}}{2} \tag{11}$$

the real part being pure gradient:

$$a_\mu = \rho_{,\mu}. \tag{12}$$

Then by substitution

$$\psi = \phi e^{-\rho}. \tag{13}$$

Equation (7) is reduced to the form

$$-g^{\mu\nu}(\nabla_\mu - i b_\mu)(\nabla_\nu - i b_\nu)\Psi + U\Psi = E\Psi. \tag{14}$$

The quantities b_μ determine the field tensor $F_{\mu\nu} = b_{\nu,\mu} - b_{\mu,\nu}$ which is responsible for the presence of a magnetic field, in general $F_{\mu\nu} \neq 0$.

This is not the end of the story, however. It follows from (9)–(11) that the potential is, generally speaking, complex:

$$U = U_1 + iU_2$$
$$U_1 = g^{\mu\nu} a_\mu a_\nu - \frac{(\sqrt{g} a^\mu)_{,\mu}}{\sqrt{g}} - g^{\mu\nu} b_\mu b_\nu \tag{15}$$
$$U_2 = 2a_\mu b^\mu - \frac{(\sqrt{g} b^\mu)_{,\mu}}{\sqrt{g}}.$$

For the equation under consideration to have the meaning of the Schrödinger equation it is necessary that $U_2 = 0$.

Until now we have not used the concrete form of coefficients $g^{\mu\nu}$ and T^μ. Now substitute (5) and (6) into (10). Then the real part of this equation gives us

$$C_{ab} h^{a\mu} \left(2h^{b\nu} a_\nu - h^{b\nu} \frac{\sqrt{g}_{,\nu}}{\sqrt{g}} - h^{b\nu}_{,\nu}\right) = 0. \tag{16}$$

Now invoke an additional assumption [13]: let operators L^a be Hermitian in some metric $g^{(0)}_{\mu\nu}$ in which the scalar product is determined in a standard way:

$$(\phi_2, \phi_1) = \int \mathrm{d}^{n-1}x \sqrt{g^{(0)}} \phi_2^* \phi_1$$
$$g^{(0)} = \det g_{\mu\nu}. \tag{17}$$

For example, for the $SO(n)$ group $g^{(0)}_{\mu\nu}$ is the metric of a $n-1$ dimensional hypersphere. Then the hermiticity condition

$$(\phi_2, L^a \phi_1) = (L^a \phi_2, \phi_1) \tag{18}$$

along with (2) and (17) entails

$$h^{a\mu}_{,\mu} = -h^{a\mu}\frac{\sqrt{g^{(0)}}_{,\mu}}{\sqrt{g^{(0)}}}. \tag{19}$$

Substitute (19) into (16) and obtain:

$$C_{ab}h^{a\mu}h^{b\nu}\left(2a_\nu - \frac{\sqrt{g}_{,\nu}}{\sqrt{g}} + \frac{\sqrt{g^{(0)}}_{,\nu}}{\sqrt{g^{(0)}}}\right) = 0. \tag{20}$$

It is clear that irrespective of C_{ab} there exists the solution [13]

$$a_\mu = \frac{\sqrt{g}_{,\mu}}{\sqrt{g}} - \frac{\sqrt{g^{(0)}}_{,\mu}}{\sqrt{g^{(0)}}} \tag{21}$$

which obeys the condition (12) of integrability and

$$\rho = \frac{1}{2}\ln\sqrt{\frac{g}{g^{(0)}}}. \tag{22}$$

We have from (13), (22):

$$(\Psi, \Psi) = \int \mathrm{d}^{n-1}x\sqrt{g}|\Psi|^2 = \int \mathrm{d}^{n-1}x\sqrt{g^{(0)}}|\phi|^2. \tag{23}$$

Thus, normalization of Ψ in the space with the metric $g_{\mu\nu}$ is determined by normalization of ϕ in the space with the metric $g^{(0)}_{\mu\nu}$, so normalizability of ϕ entails normalizability of Ψ.

Until now our treatment has run almost along the same lines as in [13] where it was assumed $C_a = 0$. The key new moment which makes our problem non-trivial is that for $C_a \neq 0$ the integrability condition (12) is not, generally speaking, sufficient for hermiticity of Hamiltonian of the Schrödinger equation (14) since it contains the term iU_2.

Now substitute (6), (11) and (21) into (15). Then we have

$$2a_\mu b^\mu = \frac{1}{2}h^{a\mu}C_a\left(\frac{\sqrt{g}_{,\mu}}{\sqrt{g}} - \frac{\sqrt{g^{(0)}}_{,\mu}}{\sqrt{g^{(0)}}}\right) \tag{24}$$

$$\frac{(b^\mu\sqrt{g})_{,\mu}}{\sqrt{g}} = b^\mu_{,\mu} + \frac{\sqrt{g}_{,\mu}}{\sqrt{g}}b^\mu = \frac{C_a}{2}\left(h^{a\mu}_{,\mu} + \frac{\sqrt{g}_{,\mu}}{\sqrt{g}}h^{a\mu}\right). \tag{25}$$

Now, again making use of (19), we see that expressions (24) and (25) coincide completely, so according to (15) $U_2 = 0$!

Thus, the general form of generators (2) along with the hermiticity condition (17) and (18) entail integrability condition (12) and, simultaneously, ensure that the potential is real. In other words, if generators (2) Hermitian in a space with the metric $g^{(0)}_{\mu\nu}$ Hamiltonian of the Schrödinger equation, (14) is Hermitian in a space with the metric $g_{\mu\nu}$.

The result obtained shows that there is essential difference between QESM based on $SU(n)$ groups and those based on $SO(n)$ ones. In the first case even when a magnetic field is absent it is a rather difficult task to find coefficients C_{ab} for which the integrability condition (12) is satisfied. In the second case we obtain at once QESM with well defined

Hamiltonian for which this condition is satisfied automatically and, moreover, the effective magnetic field is present.

As an example we will discuss briefly QESM generated by the $SO(4)$ group. Six generators of this group $J_{ik} = -i(x^i(\partial/\partial x^k) - x^k(\partial/\partial x^i)$ can be written down in hyperspherical coordinates $x^1 = \zeta\sin\theta\sin\xi\cos\varphi$, $x^2 = \zeta\sin\theta\sin\xi\sin\varphi$, $x^3 = \zeta\sin\theta\cos\xi$, $x^4 = \zeta\cos\theta$ in the form $J_{ik} = -iL_{ik}$

$$\begin{aligned}
L_{12} &= \frac{\partial}{\partial\varphi} \qquad L_{32} = \sin\varphi\frac{\partial}{\partial\xi} + \cos\varphi\,\mathrm{cotan}\,\xi\frac{\partial}{\partial\varphi} \\
L_{31} &= \cos\varphi\frac{\partial}{\partial\xi} - \sin\varphi\,\mathrm{cotan}\,\xi\frac{\partial}{\partial\varphi} \\
L_{41} &= \sin\xi\cos\varphi\frac{\partial}{\partial\theta} + \mathrm{cotan}\,\theta\cos\varphi\cos\xi\frac{\partial}{\partial\xi} - \frac{\mathrm{cotan}\,\theta\sin\varphi}{\sin\xi}\frac{\partial}{\partial\varphi} \\
L_{42} &= \sin\xi\sin\varphi\frac{\partial}{\partial\theta} + \mathrm{cotan}\,\theta\sin\varphi\cos\xi\frac{\partial}{\partial\xi} + \frac{\mathrm{cotan}\,\theta\cos\varphi}{\sin\xi}\frac{\partial}{\partial\varphi}.
\end{aligned} \tag{26}$$

For Hamiltonian (1) the metric (5), potential (15) and field tensor $F_{\mu\nu}$ are, in general, very cumbersome. Detailed analysis of the manifold generated by the group in question as well as the quantum dynamics of a particle in it could be a subject for separate investigation. Here we will consider only two simplest examples as illustrations.

Let the Hamiltonian have the form

$$H = J_{12}^2 + J_{32}^2 + J_{31}^2 + J_{41}^2 + J_{42}^2 + J_{43}^2 + CJ_{12}. \tag{27}$$

Then the metric (5) of the manifold is

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\xi^2 + \sin^2\theta\sin^2\xi\,\mathrm{d}\varphi^2. \tag{28}$$

The potential and field tensor equal

$$U = -\frac{C^2}{2}\sin^2\xi\sin^2\theta \tag{29}$$

$$\begin{aligned}
F_{\theta\varphi} &= -C\sin^2\xi\sin\theta\cos\theta \\
F_{\xi\varphi} &= -C\sin^2\theta\sin\xi\cos\xi \\
F_{\xi\theta} &= 0.
\end{aligned} \tag{30}$$

In other words, we get the Schrödinger equation for a particle moving on a hypersphere under the influence of the potential (29) and a magnetic field described by the tensor (30). In this case variables can be separated and the problem even proves to be exactly solvable, its solutions being generalized spherical harmonics.

The more complicated example is Hamiltonian

$$H = J_{41}^2 + J_{42}^2 + J_{43}^2 + CJ_{32}. \tag{31}$$

Now the metric (5) takes the form

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{\mathrm{d}R^2}{(1+R^2)^2} + R^2(\mathrm{d}\xi^2 + \sin^2\xi\,\mathrm{d}\varphi^2) \\
R &= \tan\theta.
\end{aligned} \tag{32}$$

The potential (15) and field tensor $F_{\mu\nu}$ equal

$$U = -\left[1 + \frac{2}{1+R^2} + \frac{C^2R^2}{4}(1 - \cos^2\varphi\sin^2\xi)\right] \tag{33}$$

$$\begin{aligned} F_{R\varphi} &= -CR\sin\xi\cos\xi\cos\varphi \\ F_{\xi R} &= CR\sin\varphi \\ F_{\xi\varphi} &= CR^2\sin^2\xi\cos\varphi. \end{aligned} \tag{34}$$

The manifold is non-compact: its three-volume $\int d^3x\sqrt{g} = \infty$.

It is worth noting that as a matter of fact the known cases when the Schrödinger equation with a magnetic field is exactly solvable are exhausted by a harmonic oscillator [14] and a free particle (see any textbook) in an homogeneous field. Being exactly solvable, the problem in question can be formally thought of as the particular case of QESM according to the general relation between exactly and quasi-exactly solvable models (see [7], section 4i). On the other hand, the approach developed in the present letter enables one to obtain QESM with a magnetic field which is not reduced to exactly solvable models. In so doing, one obtains many-parametric classes of solutions at once. Whereas exact solutions in a homogeneous magnetic field are found by separation of variables, the exact solutions under discussion exist in general without separation of variables. Of particular interest is the three-dimensional case when one can speak about a magnetic field literally.

The inevitable price for the possibilities indicated is curvature of manifold. However, this can be of interest on its own and useful for applications, for example, in relativistic cosmology. In particular, the metric (28) describes geometry of the Einstein universe (see, for example, [15], ch 10).

An interesting problem worth considering is to generalize the approach of this paper to non-Abelian gauge fields.

References

[1] Razavy M 1980 *Am. J. Phys.* **48** 285
[2] Razavy M 1981 *Phys. Lett.* **82A** 7
[3] Zaslavskii O B, Ulyanov V V and Tsukernik V M 1983 *Fiz. Nizk. Temp.* **9** 511; 1983 *Sov. J. Low Temp. Phys.* **9** 259
[4] Zaslavskii O B and Ulyanov V V 1984 *Zh. Eksp. Teor. Fiz.* **87** 1724; 1984 *Sov. Phys.–JETP* **60** 991
[5] Zaslavskii O B and Ulyanov V V 1987 *Teor. Mat. Fiz.* **71** 260; 1987 *Theor. Math. Phys. USSR* **71** 520
[6] Turbiner A V 1988 *Commun. Math. Phys.* **118** 467
[7] Shifman M A 1989 *Int. J. Mod. Phys.* A **4** 2897
[8] Ushveridze A G 1989 *Sov. J. Part. Nucl.* **20** 504
[9] Ulyanov V V and Zaslavskii O B 1992 *Phys. Rep.* **216** 179
[10] Shifman M A and Turbiner A V 1989 *Commun. Math. Phys.* **126** 347
[11] Zaslavskii O B 1993 *J. Phys. A: Math. Gen.* **26** 6563
[12] González-López A, Kamran N and Olver P J 1992 New quasi-exactly solvable Hamiltonians in two dimensions *Preprint* School of Math, University of Minnesota; 1994 *Commun. Math. Phys.* **159** 503
[13] Morozov A Yu, Perelomov A M, Rosly A A, Shifman M A and Turbiner A V 1990 *Int. J. Mod. Phys.* A **5** 803
[14] Fock V A 1928 Z. *Phys.* Bd **49** H7–8, S507
[15] Tolman R C 1969 *Relativisity, Thermodynamics and Cosmology* (Oxford: Clarendon)